Analytical expressions for the strong field dependence of the effective fine structure constant

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Abstract. We provide analytical and non-perturbative expressions for the effective coupling constant of QED in the presence of slowly varying background fields. Our results agree with previous numerical calculations but, for strong magnetic fields, we observe some deviations from the expected logarithmic increase of the fine structure constant. These effects tend to reduce the effective charge, thereby providing further evidence against the existence of a new, strong-coupling phase of QED in heavy-ion collisions.

Introduction

The fine structure constant can be regarded as an effective coupling constant α_{eff} which receives corrections in the presence of external electromagnetic fields [1]. By properly choosing the background field configuration and strength one can hope to shift the value of α_{eff} up to the strong coupling regime $\alpha_{\text{eff}} \sim 1$, where QED is supposed to have a new confining phase [2]. The existence of such a phase in heavy-ion collisions has been postulated [3, 4] in order to explain the narrow e^+e^- peaks observed at GSI by the EPOS and ORANGE collaborations [5, 6].

The 'proper time' formalism of Schwinger [7] has been used in [1] to estimate α_{eff} as a function of constant background fields. The numerical results show that the effective charge increases but the growth is only logarithmic and it is not enough to trigger the postulated phase transition.

The aim of this paper is to provide analytical expressions for the α_{eff} dependence on the external field strength and direction. Our results will confirm the numerical analysis of [1] but, for strong magnetic fields, we find some deviations from the expected logarithmic growth. Such effects conspire to make α_{eff} smaller, so that it is even more difficult to reach the critical point $\alpha_{\text{eff}} \sim 1$.

2. An effective Lagrangian for QED in the presence of background fields

In this section we describe in some detail the method used to evaluate the effective fine structure constant as a function of constant background fields. We start from the generating functional for QED:

$$W[J] = \int \mathscr{D}A_\mu \exp\left\{ i \int d^4x [\mathscr{L}_A + J_\mu A^\mu] \right\}$$

$$\times \int \mathscr{D}\bar{\psi}\, \mathscr{D}\psi \exp\left\{ i \int d^4x [\mathscr{L}_f + e\bar{\psi}\gamma_\mu \psi A^\mu] \right\} \tag{1}$$

where $\mathscr{L}_A$ and $\mathscr{L}_f$ are the free field Lagrangians for photons and fermions respectively. Formally, one can integrate over the fermion variables to obtain a generating functional for the photon field only:

$$W_A[J] = \int \mathscr{D}A_\mu \exp\left\{ i \int d^4x [\mathscr{L}_A + \mathscr{L}' + J_\mu A^\mu] \right\} \tag{2}$$

with

$$\exp\left\{ i \int d^4x\, \mathscr{L}' \right\} = \frac{\int \mathscr{D}\bar{\psi}\, \mathscr{D}\psi\, \exp\{i \int d^4x [\mathscr{L}_f + e\bar{\psi}\gamma_\mu \psi A^\mu]\}}{\int \mathscr{D}\bar{\psi}\, \mathscr{D}\psi\, \exp\{i \int d^4x\, \mathscr{L}_f\}}. \tag{3}$$

The electromagnetic field can now be regarded as a closed system governed by the Lagrangian $\mathscr{L} = \mathscr{L}_A + \mathscr{L}'$. $\mathscr{L}'$ includes, in an effective way, the dynamics of the fermion fields. Graphically, it represents the contributions with a single fermion loop and an arbitrary number of external photon legs. Schwinger [7] provided an evaluation of $\mathscr{L}'$ for the case of slowly varying fields $F_{\mu\nu}$. In particular $\mathscr{L}'$ can be given the following integral representation:

$$\mathscr{L}' = \frac{m^4}{8\pi^2} \int_0^\infty d\eta\, \frac{e^{-\eta}}{\eta^3} \left[-(\eta a \cot(\eta a)\eta b \coth(\eta b)) + 1 - \frac{\eta^2}{3}(a^2 - b^2) \right] \tag{4}$$

with

$$a = \frac{e}{m^2} [(\mathscr{F}^2 + \mathscr{G}^2)^{1/2} + \mathscr{F}]^{1/2} \tag{5a}$$

$$b = \frac{e}{m^2} [(\mathscr{F}^2 + \mathscr{G}^2)^{1/2} - \mathscr{F}]^{1/2} \tag{5b}$$

where $\mathscr{F}$ and $\mathscr{G}$ are the fundamental invariants of the electromagnetic field, $\mathscr{F} = (E^2 - B^2)/2$, $\mathscr{G} = E \cdot B$. If the vectors E and B are mutually parallel the invariants a and b have a simple physical meaning, namely

$$a = \frac{eE}{m^2} \qquad b = \frac{eB}{m^2}. \tag{5c}$$

Since we want to discuss a quantized electromagnetic field A_μ^q in the presence of an external classical field A_μ^{ext}, we find it appropriate to write $A_\mu = A_\mu^q + A_\mu^{ext}$ and to expand $\mathscr{L}'$ in powers of A_μ^q. The quadratic term

$$\mathscr{L}_{eff} = \frac{1}{2} \frac{\partial \mathscr{L}'}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)} \partial_\alpha A_\beta^q \partial_\gamma A_\delta^q \equiv \Lambda^{\alpha\beta\gamma\delta} \partial_\alpha A_\beta^q \partial_\gamma A_\delta^q \tag{6}$$

is then used as an effective interaction Lagrangian in computing the corrections to the photon propagator. As one can easily verify, the tensor $\Lambda^{\alpha\beta\gamma\delta}$ has the following symmetry properties:

$$\Lambda^{\alpha\beta\gamma\delta} = -\Lambda^{\beta\alpha\gamma\delta} = -\Lambda^{\alpha\beta\delta\gamma} = \Lambda^{\gamma\delta\alpha\beta}. \tag{7}$$

In terms of this tensor the leading correction to the photon propagator is given by

$$\delta D_{\mu\nu}(x-y) = -\int d^4z \langle 0| T A_\mu(x) A_\nu(y) \Lambda^{\alpha\beta\gamma\delta} [\partial_\alpha A_\beta(z)][\partial_\gamma A_\delta(z)]|0\rangle \tag{8}$$

where the superscript 'q' has been omitted. From the corrected propagator in momentum space $\mathscr{D}_{\mu\nu}(k)$ we extract the effective fine structure constant by means of the gauge invariant definition:

$$\alpha J_1^\mu \mathscr{D}_{\mu\nu}(k) J_2^\nu \equiv \alpha_{\text{eff}} J_1^\mu D_{\mu\nu}(k) J_2^\nu \tag{9a}$$

where the matrix elements of the currents $J_{1,2}$ are taken in their static limit $J_{1,2} \to (\rho_{1,2}, 0)$ and, consequently, $k \to (0, \boldsymbol{k})$. More conveniently, α_{eff} can be defined as

$$\alpha \mathscr{D}_{00}(0, \boldsymbol{k}) \equiv \alpha_{\text{eff}} D_{00}(0, \boldsymbol{k}) \tag{9b}$$

which is equivalent to (9a) provided that the propagators $D_{\mu\nu}(k)$ and $\mathscr{D}_{\mu\nu}(k)$ are non-singular in the considered limit $k \to (0, \boldsymbol{k})$. In the following, we shall explicitly verify that this condition is actually fulfilled if one starts from the free propagator in a standard gauge, such as the Feynman one. Strictly speaking, our α_{eff} cannot be considered as a 'true' coupling constant, since it depends on the direction of the exchanged momentum $\boldsymbol{k}$. In principle, alternative definitions are possible. In particular, following the authors of [1], α_{eff} can be given in terms of the fundamental invariants ($\mathscr{F}$, $\mathscr{G}$), providing a coupling which is automatically independent of the direction of the applied external field. However, we find it appropriate to keep definition (9) since the existence of preferred directions is an essential feature of the problem we are dealing with and, in such conditions, any isotropic quantity is likely to give an incomplete description of QED. Besides this, the isotropic fine structure constant of [1] can easily be derived from our α_{eff}, as we shall discuss in section (4).

Before considering some specific configurations of background fields it is useful to give equation (8) a more compact form. First, we apply the Wick theorem to obtain

$$\delta D_{\mu\nu}(x-y) = -\int d^4z \langle 0|TA_\mu(x)\partial_\alpha^z A_\beta(z)|0\rangle \langle 0|TA_\nu(y)\partial_\gamma^z A_\delta(z)|0\rangle \Lambda^{\alpha\beta\gamma\delta}$$

$$- \int d^4z \langle 0|TA_\mu(x)\partial_\gamma^z A_\delta(z)|0\rangle \langle 0|TA_\nu(y)\partial_\alpha^z A_\beta(z)|0\rangle \Lambda^{\alpha\beta\gamma\delta}. \tag{10}$$

Because of the symmetry properties of $\Lambda^{\alpha\beta\gamma\delta}$ the two terms of the RHS are equal, so that

$$\delta D_{\mu\nu}(x-y) = 2 \int d^4z [\partial_\alpha^z D_{\mu\beta}(x-z)][\partial_\gamma^z D_{\nu\delta}(y-z)] \Lambda^{\alpha\beta\gamma\delta}. \tag{11}$$

Finally, by introducing the Fourier transforms of the free photon propagators we obtain, in momentum space,

$$\delta D_{\mu\nu}(k) = 2 D_{\mu\beta}(k) \Lambda^{\alpha\beta\gamma\delta} k_\alpha k_\gamma D_{\delta\nu}(k). \tag{12}$$

It is important to observe that the correction $\delta D_{\mu\nu}(k)$ is invariant under the gauge transformation $D_{\mu\nu}(k) \to D_{\mu\nu}(k) + \chi_\mu k_\nu + \chi_\nu k_\mu$. This property follows from the symmetry relation (7) which ensures that $k_\beta k_\alpha \Lambda^{\alpha\beta\gamma\delta} = \Lambda^{\alpha\beta\gamma\delta} k_\gamma k_\delta = 0$.

3. Weak fields

We now use equation (12) to find α_{eff} as a function of weak background fields. We treat this problem just for the sake of completeness, since it is not strictly connected to the 'new phase scenario' proposed in [3, 4].

For weak fields, that is for E, $B \ll m^2/e$, the Lagrangian $\mathscr{L}'$ is well approximated [8] by the expression

$$\mathscr{L}' = \frac{\alpha^2}{45 \cdot 8\pi^2 m^4}[(E^2 - B^2)^2 + 7(E \cdot B)^2]. \tag{13}$$

The corresponding tensor $\Lambda^{\alpha\beta\gamma\delta}$ turns out to be

$$\Lambda^{\alpha\beta\gamma\delta} = \frac{\alpha^2}{45 \cdot 8\pi^2 m^4}[4F^{\gamma\delta}F^{\alpha\beta} + F^{lm}F_{lm}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})$$
$$+ 7\tilde{F}^{\gamma\delta}\tilde{F}^{\alpha\beta} + \tfrac{7}{4}\varepsilon^{\alpha\beta\gamma\delta}\tilde{F}^{lm}F_{lm}] \tag{14}$$

where $\tilde{F}_{lm}$ is the dual of F_{lm}. From equations (12) and (14) we get the following result for $\delta D_{\mu\nu}(k)$:

$$\delta D_{\mu\nu}(k) = \frac{(4\pi\alpha)^2}{45 \cdot 8\pi^2 m^4}\left[8\frac{F_{\alpha\mu}k^\alpha F_{\gamma\nu}k^\gamma}{(k^2)^2} + 14\frac{\tilde{F}_{\alpha\mu}k^\alpha \tilde{F}_{\gamma\nu}k^\gamma}{(k^2)^2} + 2F_{lm}F^{lm}\frac{g_{\mu\nu}}{k^2}\right]. \tag{15}$$

Obviously, the term with the completely antisymmetric tensor $\varepsilon^{\alpha\beta\gamma\delta}$ does not contribute to the photon propagator. The expression for $\delta D_{00}(k)$ is much simpler,

$$\delta D_{00}(k) = \frac{4\alpha^2}{45 m^4}\left[4\frac{(k \cdot E)^2}{(k^2)^2} + 7\frac{(k \cdot B)^2}{(k^2)^2} + \frac{2(B^2 - E^2)}{k^2}\right] \tag{16}$$

and it is straightforward to obtain the effective fine structure constant α_{eff}:

$$\alpha_{\text{eff}} = \alpha\left[1 + \frac{\alpha^2}{45\pi m^4}[2B^2 - 7(t \cdot B)^2 - 2E^2 - 4(t \cdot E)^2]\right] \tag{17}$$

where t is the unit vector in the direction of k. As far as weak magnetic fields are concerned, we see that the effective coupling strongly depends on the direction t of the exchanged momentum k: α_{eff} increases if $t \cdot B \sim 0$ but the opposite holds when the $t \sim B/B$ direction is considered. The effect of a weak electric field is easier to comprehend [1] and, as one would expect, it tends to reduce the effective fine structure constant.

4. Strong fields

Let us now consider the most interesting case of strong background fields. The knowledge of α_{eff} as a function of constant electric fields provides information about the behaviour of QED around static configurations of highly charged sources. On the contrary, the study of α_{eff} in strong magnetic fields can be regarded as a first step towards the description of QED in the neighbourhood of large moving charges, such as the heavy ions colliding in a GSI experiment.

When the electric field is so strong that $eE/m^2 \gg 1$, the formalism developed in section (2) is no longer reliable; electron-positron pairs are produced with sizeable probability and we cannot describe the dynamics of A_μ by means of a real effective Lagrangian depending on $F_{\mu\nu}$ only. Thus, we shall evaluate α_{eff} as a function of a strong magnetic field B, whereas E will still be supposed to be weak. In a heavy-ion collision the condition $E \ll B$ is actually fulfilled in the region between the colliding nuclei. The size L of this region depends on the impact parameter s of the collision as well as on the β of the ions. A rough estimate gives $L \sim \beta s$, as one can verify by means of very simple arguments.

As in the previous section, we have to start from the tensor $\Lambda^{\alpha\beta\gamma\delta}$, which can be written as

$$
\Lambda^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\frac{\partial\mathscr{L}'}{\partial a^2}\frac{\partial^2 a^2}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)} + \frac{\partial^2\mathscr{L}'}{\partial(a^2)^2}\frac{\partial a^2}{\partial(\partial_\alpha A_\beta)}\frac{\partial a^2}{\partial(\partial_\gamma A_\delta)}\right.
$$
$$
+\frac{\partial\mathscr{L}'}{\partial b^2}\frac{\partial^2 b^2}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)} + \frac{\partial^2\mathscr{L}'}{\partial(b^2)^2}\frac{\partial b^2}{\partial(\partial_\alpha A_\beta)}\frac{\partial b^2}{\partial(\partial_\gamma A_\delta)}
$$
$$
\left.+\frac{\partial^2\mathscr{L}'}{\partial a^2\partial b^2}\left(\frac{\partial a^2}{\partial(\partial_\alpha A_\beta)}\frac{\partial b^2}{\partial(\partial_\gamma A_\delta)} + \frac{\partial b^2}{\partial(\partial_\alpha A_\beta)}\frac{\partial a^2}{\partial(\partial_\gamma A_\delta)}\right)\right]. \tag{18}
$$

since the external electric field is supposed to be weak, we evaluate the derivatives of the invariants a^2 and b^2 up to the quadratic terms in E only. We obtain

$$
\frac{\partial a^2}{\partial(\partial_\alpha A_\beta)} = 2\frac{\alpha}{m^4}\left[\frac{E_\parallel^2}{B^2}F^{\beta\alpha} + \frac{E\cdot B}{B^2}\tilde{F}^{\beta\alpha}\right] \tag{19a}
$$

$$
\frac{\partial b^2}{\partial(\partial_\alpha A_\beta)} = \frac{\partial a^2}{\partial(\partial_\alpha A_\beta)} - 2\frac{\alpha}{m^4}F^{\beta\alpha} \tag{19b}
$$

$$
\frac{\partial^2 a^2}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)}
$$
$$
= \frac{\alpha}{m^4}\left[\frac{2}{B^2}\left(1 - 6\frac{E_\parallel^2}{B^2} + \frac{E^2}{B^2}\right)\tilde{F}^{\alpha\beta}\tilde{F}^{\gamma\delta}\right.
$$
$$
+8\frac{E_\parallel^2}{B^4}F^{\alpha\beta}F^{\gamma\delta} + 2\frac{E_\parallel^2}{B^2}(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta})
$$
$$
\left.+4\frac{E\cdot B}{B^4}(F^{\alpha\beta}\tilde{F}^{\gamma\delta} + F^{\gamma\delta}\tilde{F}^{\alpha\beta}) - 2\frac{E\cdot B}{B^2}\varepsilon^{\alpha\beta\gamma\delta}\right] \tag{20a}
$$

$$
\frac{\partial^2 b^2}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)} = \frac{\partial^2 a^2}{\partial(\partial_\alpha A_\beta)\partial(\partial_\gamma A_\delta)} - 2\frac{\alpha}{m^4}(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta}) \tag{20b}
$$

where $E_\parallel$ is the component of E parallel to B. As far as the derivatives of $\mathscr{L}'$ are concerned it is useful to express the Lagrangian as

$$
\mathscr{L}'(a, b) = \mathscr{L}_0(b) + a^2\mathscr{L}_2(b) + a^4\mathscr{L}_4(b) + \ldots \tag{21}
$$

where $\mathscr{L}_0$, $\mathscr{L}_2$ and $\mathscr{L}_4$ are obtained by expanding the RHS of equation (4):

$$
\mathscr{L}_0 = \frac{m^4}{8\pi^2}\int_0^\infty d\eta\,\frac{e^{-\eta}}{\eta^3}\left[\eta^2\frac{b^2}{3} - \eta b\coth(\eta b) + 1\right] \tag{22a}
$$

$$
\mathscr{L}_2 = \frac{m^4}{8\pi^2}\int_0^\infty d\eta\,\frac{e^{-\eta}}{3\eta}[\eta b\coth(\eta b) - 1] \tag{22b}
$$

$$
\mathscr{L}_4 = \frac{m^4}{8\pi^2}\frac{b}{45}\int_0^\infty d\eta\,e^{-\eta}\,\eta^2\coth(\eta b). \tag{22c}
$$

Then, keeping in mind that $a^2 = (\alpha/m^4)E_\parallel^2 + O(E^4)$, we have

$$
\frac{\partial\mathscr{L}'}{\partial a^2} = \mathscr{L}_2(b) + 2\frac{\alpha}{m^4}E_\parallel^2\mathscr{L}_4(b) + O(E^4) \tag{23a}
$$

$$
\frac{\partial\mathscr{L}'}{\partial b^2} = \frac{\partial\mathscr{L}_0'}{\partial b^2} + \frac{\alpha}{m^4}E_\parallel^2\frac{\partial\mathscr{L}'}{\partial b^2} + O(E^4) \tag{23b}
$$

and similar relations for the higher order derivatives. All the previous expressions are greatly simplified if we set $E = 0$. In particular, the tensor $\Lambda^{\alpha\beta\gamma\delta}$ takes the form

$$\Lambda^{\alpha\beta\gamma\delta} = \frac{\alpha}{m^4} \left[\left(\frac{\partial \mathscr{L}'}{\partial a^2} + \frac{\partial \mathscr{L}'}{\partial b^2} \right) \frac{\tilde{F}^{\alpha\beta}\tilde{F}^{\gamma\delta}}{B^2} - \frac{\partial \mathscr{L}'}{\partial b^2} (g^{\beta\gamma}g^{\alpha\delta} - g^{\alpha\gamma}g^{\beta\delta}) \right.$$

$$\left. + 2 \frac{\alpha}{m^4} \frac{\partial^2 \mathscr{L}'}{\partial (b^2)^2} F^{\alpha\beta}F^{\gamma\delta} \right] \tag{24a}$$

$$= \frac{\alpha}{m^4} \left[\left(\mathscr{L}_2 + \frac{\partial \mathscr{L}_0}{\partial b^2} \right) \frac{\tilde{F}^{\alpha\beta}\tilde{F}^{\gamma\delta}}{B^2} - \frac{\partial \mathscr{L}_0}{\partial b^2} (g^{\beta\gamma}g^{\alpha\delta} - g^{\alpha\gamma}g^{\beta\delta}) \right.$$

$$\left. + 2 \frac{\alpha}{m^4} \frac{\partial^2 \mathscr{L}_0}{\partial (b^2)^2} F^{\alpha\beta}F^{\gamma\delta} \right]. \tag{24b}$$

According to equation (12), the corresponding correction to the photon propagator turns out to be

$$\delta D_{\mu\nu}(k) = 2 \frac{\alpha}{m^4} (4\pi)^2 \left[\frac{1}{B^2} \left(\mathscr{L}_2 + \frac{\partial \mathscr{L}_0}{\partial b^2} \right) \frac{\tilde{F}_{\alpha\mu}k^\alpha \tilde{F}_{\gamma\nu}k^\gamma}{(k^2)^2} + \frac{\partial \mathscr{L}_0}{\partial b^2} \frac{g_{\mu\nu}}{k^2} \right.$$

$$\left. + 2 \frac{\alpha}{m^4} \frac{\partial^2 \mathscr{L}_2}{\partial (b^2)^2} \frac{F_{\alpha\mu}k^\alpha F_{\gamma\nu}k^\gamma}{(k^2)^2} - \frac{\partial \mathscr{L}_0}{\partial b^2} \frac{k_\mu k_\nu}{k^2} \right]. \tag{25}$$

Once again the correction to the $D_{00}(k)$ component has a rather simple expression:

$$\delta D_{00}(0, k) = \frac{4\pi}{k^2} \frac{8\pi\alpha}{m^4} \left[\frac{\partial \mathscr{L}_0}{\partial b^2} + \left(\mathscr{L}_2 + \frac{\partial \mathscr{L}_0}{\partial b^2} \right) \frac{(k \cdot B)^2}{k^2 B^2} \right]. \tag{26}$$

From this we extract the effective coupling constant α_{eff}:

$$\alpha_{\text{eff}} = \alpha \left\{ 1 + \frac{8\pi\alpha}{m^4} \left[\frac{\partial \mathscr{L}_0}{\partial b^2} - \left(\mathscr{L}_2 + \frac{\partial \mathscr{L}_0}{\partial b^2} \right) \frac{(t \cdot B)^2}{B^2} \right] \right\}. \tag{27}$$

The problem is now reduced to find the asymptotic behaviour, for $b \to \infty$, of $\partial \mathscr{L}_0 / \partial b^2$ and $\mathscr{L}_2(b)$. Starting from equations (22a,b) it is not difficult to recognize that

$$\frac{\partial \mathscr{L}_0}{\partial b^2} \sim \frac{m^4}{8\pi^2} \frac{1}{6} \ln b^2 \equiv \frac{m^4}{8\pi^2} \frac{1}{6} \ln \left(\frac{\alpha B^2}{m^4} \right) \tag{28}$$

$$\mathscr{L}_2 \sim \frac{m^4}{8\pi^2} \left(\frac{b}{3} - \frac{1}{6} \ln b^2 \right) \equiv \frac{m^4}{8\pi^2} \left[\frac{eB}{3m^2} - \frac{1}{6} \ln \left(\frac{\alpha B^2}{m^4} \right) \right]. \tag{29}$$

Later on we shall need the asymptotic behaviour of $\mathscr{L}_4$ as well. The integral appearing in equation (22c) can be evaluated analytically and it gives

$$\int_0^\infty d\eta \, e^{-\eta} \eta^2 \coth(\eta b) = 2 \left[\frac{1}{4b^3} \zeta \left(3, \frac{1}{2b} \right) - 1 \right] \tag{30}$$

where $\zeta(n, x)$ is the generalized Riemann zeta function. From this result one easily obtains

$$\mathscr{L}_4(b) \sim \frac{2}{45} \frac{m^4}{8\pi^2} b. \tag{31}$$

Inserting the relations (28) and (29) in equation (27) we get

$$\alpha_{\text{eff}} \sim \alpha \left[1 + \frac{\alpha}{6\pi} \ln \left(\frac{\alpha B^2}{m^4} \right) - \frac{\alpha}{3\pi} \frac{eB}{m^2} \frac{(t \cdot B)^2}{B^2} \right]. \tag{32}$$

This formula looks very strange since it predicts a negative α_{eff} when $t \cdot B \neq 0$ and $B \gg (3\pi/\alpha)(m^2/e)$. The origin of this drawback is easily recognized: equation (32) has been obtained from the first-order correction (12) which is not enough to account for the big corrections induced by strong external fields. In order to obtain meaningful expressions we have then to evaluate the photon propagator to all orders in $\mathscr{L}_{\text{eff}}$. This 'exact' propagator $\mathscr{D}$ can be cast into the form

$$\mathscr{D} = D + DVD + DVDVD + \ldots = D[D - DVD]^{-1}D = D[D - \delta D]^{-1}D \tag{33}$$

where V is the external field vertex operator $V^{\beta\delta} = 2k_\alpha k_\gamma \Lambda^{\alpha\beta\gamma\delta}$. If we work in the Feynman gauge $D_{\mu\nu} = 4\pi g_{\mu\nu}/k^2$ and we set $k_0 = 0$, the matrix $D - \delta D$ has the simple block structure

$$D - \delta D \sim \begin{bmatrix} \Box & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix} \tag{34}$$

as one can easily verify with the aid of equation (25). This result enables us to write

$$\mathscr{D}_{00} = D_{00}[D - \delta D]_{00}^{-1}D_{00} = \frac{D_{00}}{1 - \delta D_{00}} \tag{35}$$

which, in turn, yields

$$\alpha_{\text{eff}} = \frac{\alpha}{1 - \delta\alpha} \tag{36}$$

where $\delta\alpha$ stands for the first-order correction to the effective fine structure constant. The use of the Feynman gauge has slightly simplified the calculations but the final result for α_{eff} does not depend on that choice. To see this, it suffices to recall that $V^{\beta\alpha}k_\alpha = k_\beta V^{\beta\alpha} = 0$ (see comments on equation (12)). Thus, if we add a gauge term $G_{\mu\nu} = \chi_\mu k_\nu + \chi_\nu k_\mu$ to $D_{\mu\nu}$, the corrected propagator given by equation (33) will change in the same manner, that is

$$\mathscr{D}_{\mu\nu} \to \mathscr{D}_{\mu\nu} + \chi_\mu k_\nu + \chi_\nu k_\mu \tag{37}$$

with no effect on equations (9).

Having established relation (36) we can now replace equation (32) with the improved one:

$$\alpha_{\text{eff}} = \alpha \left/ \left(1 - \frac{\alpha}{6\pi} \ln\left(\frac{\alpha B^2}{m^4}\right) + \frac{\alpha}{3\pi} \frac{eB}{m^2} \frac{(t \cdot B)^2}{B^2} \right) \right. \tag{38}$$

As in the case of weak magnetic fields, we observe an increase of α_{eff} in the $t \cdot B = 0$ plane. However, the growth is too small to corroborate the hypothesis according to which the critical point $\alpha_{\text{eff}} \sim 1$ is approached in a heavy-ion collision. In fact, the strength of the magnetic field produced between the colliding nuclei is of the order $B^{\text{ion}} \sim 10^3 \text{ MeV}^2$, corresponding to a relative correction $(\alpha_{\text{eff}} - \alpha)/\alpha \sim 0.005$ only. Furthermore, as soon as we move away from the $t \cdot B = 0$ plane, the leading effect of the applied field is to reduce the coupling, since the logarithmic growth is dominated by the linear term

$$\frac{\alpha}{3\pi} \frac{eB}{m^2} \frac{(t \cdot B)^2}{B^2}.$$

In this regard, it is easy to understand why such an effect was not reported in [1]. In this reference the effective fine structure constant was defined as

$$\bar{\alpha} = \frac{\alpha}{4\pi} \left(\frac{\partial \mathscr{L}}{\partial \mathscr{F}} \right)^{-1} \equiv \frac{\alpha}{4\pi} \left[\frac{\partial}{\partial \mathscr{F}} (\mathscr{L}_A + \mathscr{L}') \right]^{-1}. \tag{39}$$

Since $\mathscr{L}_A = \mathscr{F}/4\pi$ we can write

$$\bar{\alpha} = \alpha \left[1 + 4\pi \frac{\partial \mathscr{L}'}{\partial \mathscr{F}} \right]^{-1} = \alpha \left[1 - 8\pi \frac{\alpha}{m^4} \frac{\partial \mathscr{L}'}{\partial b^2} \right]^{-1}. \tag{40}$$

That is, $\bar{\alpha}$ is nothing but the value taken by our α_{eff} at $t \cdot B = 0$. Thus, its asymptotic behaviour is given by

$$\bar{\alpha} \sim \alpha \left/ \left[1 - \frac{\alpha}{6\pi} \ln \left(\frac{\alpha B^2}{m^4} \right) \right] \right. \tag{41}$$

without terms proportional to B in the denominator.

The foregoing results, in particular equation (38), require minor emendations when a weak electric field is superimposed on B. In this case it is useful to split the tensor $\Lambda^{\alpha\beta\gamma\delta}(E, B)$ as follows:

$$\Lambda^{\alpha\beta\gamma\delta}(E, B) = \Lambda^{\alpha\beta\gamma\delta}(B) + \lambda^{\alpha\beta\gamma\delta} \tag{42}$$

where $\Lambda^{\alpha\beta\gamma\delta}(B)$ is given by equation (24) and $\lambda^{\alpha\beta\gamma\delta}$ accounts for the corrections introduced by the external electric field. Correspondingly, the first-order correction to the photon propagator is written as

$$\delta D_{\mu\nu}(E, B) = \delta D_{\mu\nu}(B) + \delta W_{\mu\nu} \tag{43}$$

where $\delta D_{\mu\nu}(B)$ is given by equation (25) and

$$\delta W_{\mu\nu}(k) = 2 D_{\mu\beta}(k) \lambda^{\alpha\beta\gamma\delta} k_\alpha k_\gamma D_{\delta\nu}(k) \tag{44}$$

as dictated by equation (12). Then, according to equation (33), the 'exact' propagator $\mathscr{D}(E, B)$ takes the form

$$\mathscr{D}(E, B) = D[D - \delta D(B) - \delta W]^{-1} D \approx \mathscr{D}(B)[1 + D^{-1}\delta W D^{-1}\mathscr{D}(B)] \tag{45}$$

$\mathscr{D}(B)$ being the exact propagator in the absence of background electric fields. From this equation we obtain the following relation for $\alpha_{\text{eff}}(E, B)$:

$$\alpha_{\text{eff}}(E, B) = \alpha_{\text{eff}}(B) \left[1 + D_{00}^{-1} \delta W_{00} \frac{\alpha_{\text{eff}}(B)}{\alpha} \right] \tag{46}$$

with $\alpha_{\text{eff}}(B)$ given by equation (38). The correction δW_{00} can be evaluated with the aid of equations (19) and (20). Since in this case the tensor $\lambda^{\alpha\beta\gamma\delta}$ takes a rather messy form we just state the final result, omitting the intermediate steps. It turns out that $\alpha_{\text{eff}}(E, B)$ can be cast into the form

$$\alpha_{\text{eff}}(E, B) \sim \alpha_{\text{eff}}(B) \left[1 - \frac{4}{15\pi} \frac{eB}{m^2} \frac{\alpha^2}{m^4} E_\parallel^2 \frac{(t \cdot B)^2}{B^2} \frac{\alpha_{\text{eff}}(B)}{\alpha} \right]. \tag{47}$$

The logarithmic growth in the $t \cdot B = 0$ plane is then unaffected but, for $t \cdot B \neq 0$, the presence of a weak electric field gives a further contribution to the decrease of the effective coupling constant.

5. Summary

In this work we have discussed the photon propagator in the presence of slowly varying external fields. Analytical and non-perturbative results have been obtained by including the fermionic dynamics in an effective Lagrangian depending on the photon field only. From the $\mathcal{D}_{00}$ component of the corrected propagator we have extracted an effective fine structure constant α_{eff} as a function of the external field configuration.

The behaviour of α_{eff} in the presence of strong fields has interesting implications for the e^+e^- peaks observed in heavy-ion scattering experiments at GSI [5, 6]. More precisely, it is important to ascertain whether the unusual field environment induced by the heavy ions can shift α_{eff} up to the strong coupling regime $\alpha_{\text{eff}} \sim 1$, where QED is supposed to have a new confining phase; as suggested by many authors [3, 4], the presence of this new phase would actually explain the gross features of the observed e^+e^- narrow structures. In this regard, the main motivation for our work has been to reproduce analytically the numerical results of [1] which show that the increase of α_{eff} is too small to trigger the postulated phase transition.

From the formalism developed in sections (2) and (4) we have derived a simple expression describing the behaviour of α_{eff} in the presence of a strong magnetic field $\boldsymbol{B}$. In the plane perpendicular to $\boldsymbol{B}$ the effective coupling shows a logarithmic growth which agrees with the numerical evaluation of [1]. For a field strength $B \sim 10^3 \text{ MeV}^2$, comparable with that produced in a heavy-ion collision, we have estimated a negligible shift of the effective coupling constant, namely $\alpha_{\text{eff}} - \alpha / \alpha \sim 0.005$. This small correction clearly militates against the new phase hypothesis.

Sizeable deviations from the expected logarithmic increase have been found for directions with a non-vanishing projection on $\boldsymbol{B}$. In particular, the dominant effect of the applied field is to reduce the effective charge, thereby providing further evidence against the existence of a strong coupling phase of QED in heavy-ion collisions. Anyway, one has to keep in mind that our formalism applied to slowly varying fields only. As a consequence, we cannot rigorously rule out the possibility that a phase transition is triggered by more realistic configurations of external fields. A lattice calculation [9] shows that this is unlikely to occur for the Coulomb case, but the role played by time-dependent fields is still an open question. Moreover, further investigations are required if the Lagrangian for QED is modified from the beginning via an additional fermion-fermion contact interaction, which may be relevant to the strong coupling phase of QED. A similar question has been addressed in [10], suggesting that the new interaction term can greatly affect the behaviour of QED in strong background fields.

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